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BACKWARD EXTENSION OF SUBNORMAL 2-VARIABLE WEIGHTED SHIFTS

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Abstract. We study backward extensions of 2-variable weighted shifts with finite atomic Berger measure. We provide a necessary and sufficient condition for the subnormality of such extensions. As an application, we give a simple counterexample for the Curto-Muhly-Xia conjecture [10].

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [2], [10], [14]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

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For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty} \in \ell^\infty(\mathbb{Z}_+)$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0. \end{cases}$$

It is easy to see that $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots)$ is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{(k_1,k_2)}\}, \beta \equiv \{\beta_{(k_1,k_2)}\} \in \ell^\infty(\mathbb{Z}_+^2)$, $(k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{(k_1,k_2)} := \alpha_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_1}$$

$$T_2 e_{(k_1,k_2)} := \beta_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{(k_1,k_2)+\varepsilon_1} \alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)+\varepsilon_2} \beta_{(k_1,k_2)} \tag{1.1}$$

for all $(k_1, k_2) \in \mathbb{Z}_+^2$.

In an entirely similar way one can define multivariable weighted shifts.

Besides their relevance for the construction of examples and counterexamples in Hilbert space operator theory, weighted shifts can also be used to detect properties such as subnormality, via the Lambert-Lubin Criterion([16, 17]): *A commuting pair (T_1, T_2) of injective operators acting on a Hilbert space $\mathcal{H}$ is subnormal if and only if the 2-variable weighted shift with weights $\alpha_{(i,j)} := \frac{||T_1^{i+1} T_2^j x||}{||T_1^i T_2^j x||}$ and $\beta_{(i,j)} := \frac{||T_1^i T_2^{j+1} x||}{||T_1^i T_2^j x||}$ is subnormal for every nonzero vector $x \in \mathcal{H}$.* Thus, to study the subnormality of commuting pairs, we focus on weighted shifts in the sequel.

Given $(k_1, k_2) \in \mathbb{Z}_+^2$, the *moment* of (α, β) of order (k_1, k_2) is

(1.2)

$$\gamma_{(k_1,k_2)} := \begin{cases} 1, & \text{if } (k_1, k_2) = (0, 0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (1.1), $\gamma_{(k_1,k_2)}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

We also recall a well known characterization of subnormality for multivariable weighted shifts $\mathbf{T} \equiv (T_1, \cdots, T_n)$, due to C. Berger in the 1-variable case (cf. [13], [15]); for simplicity, we state it in the case $n = 2$: $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is subnormal if and only if there is a probability measure μ defined on the rectangle $R = [0, a_1] \times [0, a_2]$ where $a_i = \|T_i\|^2$ such that

$$\gamma_{(k_1,k_2)} = \int_R s^{k_1} t^{k_2} d\mu(s,t),$$

for all $(k_1, k_2) \in \mathbb{Z}_+^2$.

In the single variable case, if W_α is subnormal with Berger measure ξ, and if we let $\mathcal{L}_j := \bigvee\{e_{k_1} : k_1 \geq j\}$ denote the invariant subspace obtained by removing the first j vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{L}_j}$ is $d\xi_{\mathcal{L}_j}(s) = \frac{s^j}{\gamma_j} d\xi(s)$. Similarly, for an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}$ (resp. $\mathcal{N}$) be the invariant subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$).

2. Main results

We recall some auxiliary facts needed for our main results.

LEMMA 2.1. *(Subnormal backward extension of a 1-variable weighted shift) (cf [5, Proposition 8], [11, Proposition 1.5]) Let $W_\alpha|_{\mathcal{L}_1}$ be subnormal, with associated measure $\mu_{\mathcal{L}_1}$. Then W_α is subnormal (with associated measure μ) if and only if the following conditions hold:*

(i) $\frac{1}{s} \in L^1(\mu_{\mathcal{L}_1})$

(ii) $\alpha_0^2 \leq \left(\left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{L}_1})} \right)^{-1}$.

In this case,

$$d\mu(s) = \frac{\alpha_0^2}{s} d\mu_{\mathcal{L}_1}(s) + \left(1 - \alpha_0^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{L}_1})}\right) d\delta_0(s),$$

where δ_0 denotes Dirac measure at 0. In particular, W_α is never subnormal when $\mu_{\mathcal{L}_1}(\{0\}) > 0$.

To check subnormality of 2-variable weighted shifts, we introduce some definitions.

(i) Let μ and ν be two positive measures on X. We say that $\mu \le \nu$ on X, if $\mu(E) \le \nu(E)$ for all Borel subset $E \subseteq X$; equivalently, $\mu \le \nu$ if and only if $\int f d\mu \le \int f d\nu$ for all $f \in C(X)$ such that $f \ge 0$ on X.

(ii) Let μ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$.

(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that μ^X is a probability measure whenever μ is.

Then we have:

LEMMA 2.2. *(Subnormal backward extension of a* 2*-variable weighted shift* [11]*) Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with associated Berger measure $\mu_{\mathcal{M}}$ and that $shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with associated measure ξ_0. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\beta_{00}^2 \le \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}\right)^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \le \xi_0$.

Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi_0$. In the case when $W_{(\alpha,\beta)}$ is subnormal, the Berger measure μ of $W_{(\alpha,\beta)}$ is given by

(2.1)
$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + \left(d\xi_0(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)\right) d\delta_0(t).$$

A necessary condition for the subnormality of 2-variable weighted shift was known in [12]:

LEMMA 2.3. ([12, Theorem 3.1]) *Let μ be the Berger measure of a subnormal* 2*-variable weighted shift, and let ξ_j be the Berger measure of the associated j-th horizontal* 1*-variable $shift(\alpha_{0j}, \alpha_{1j}, \cdots)$ for $j \ge 0$. Similarly, let η_i be the Berger measure of the associated i-th vertical* 1*-variable $shift(\beta_{i0}, \beta_{i1}, \cdots)$ for $i \ge 0$. Finally, let $d\mu_{(i,j)}(s,t) := \frac{1}{\gamma_{(i,j)}} s^i t^j d\mu(s,t)$ (for $i, j \ge 0$). Then $\xi_j = \mu_{(0,j)}^X$ ($j \ge 0$) and $\eta_i = \mu_{(i,0)}^Y$ ($i \ge 0$).*

COROLLARY 2.4. ([12, Theorem 3.3]) *Let μ, ξ_j and η_i be as in Lemma* 2.3*. Then we have $\xi_{j+1} \ll \xi_j$ and $\eta_{i+1} \ll \eta_i$ for every $i, j \ge 0$, where $\ll$ denote the absolute continuity of measures.*

To detect the hyponormality of 2-variable weighted shifts, we use the following result.

Lemma 2.5. ([6, Theorem 2.4]) *The following statements are equivalent:*

(i) $W_{(\alpha,\beta)}$ *is hyponormal;*

(ii)

$$\begin{aligned} M_{(k_1,k_2)}(1) : &= (\gamma_{(k_1,k_2)+(m,n)+(p,q)})_{\substack{0\le m+n\le 1\\ 0\le p+q\le 1}} \\ &= \begin{pmatrix} \gamma_{(k_1,k_2)}, \gamma_{(k_1+1,k_2)}, \gamma_{(k_1,k_2+1)} \\ \gamma_{(k_1+1,k_2)}, \gamma_{(k_1+2,k_2)}, \gamma_{(k_1+1,k_2+1)} \\ \gamma_{(k_1,k_2+1)}, \gamma_{(k_1+1,k_2+1)}, \gamma_{(k_1,k_2+2)} \end{pmatrix} \ge 0 \end{aligned}$$

for all $(k_1,k_2) \in \mathbb{Z}_+^2$.

It was shown the following results in [9].

Lemma 2.6. *Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *and* $W_{(\alpha,\beta)}|_{\mathcal{N}}$ *are subnormal with associated Berger measures* $\mu_{\mathcal{M}}$ *and* $\mu_{\mathcal{N}}$*, respectively. Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if the following four conditions hold:*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;

(iii) $\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \le c_0 := \frac{\int t d\mu_{\mathcal{N}}}{\int s d\mu_{\mathcal{M}}}$;

(iv) $\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c_0 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c_0}{s} (\mu_{\mathcal{N}})^X \right\} \le \delta_0$.

Lemma 2.7. *Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *and* $W_{(\alpha,\beta)}|_{\mathcal{N}}$ *are subnormal with associated Berger measures* $\mu_{\mathcal{M}}$ *and* $\mu_{\mathcal{N}}$*, respectively. Also assume that* $\mu_{\mathcal{M}\cap\mathcal{N}} = \sigma \times \tau$ *for some* 1*-variable probability measures* σ *and* τ*. Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if the following conditions hold:*

(i) $\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \le 1$;

(ii) $\left(\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\tau)} \right) \xi_1 \le \xi_0$.

Using the Lemmas 2.2, 2.3, 2.6, 2.7, we have :

Theorem 2.8. *Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *is subnormal with* n*-atomic Berger measure* $\mu_{\mathcal{M}} = \sum_{i=0}^{n-1} \rho_i \delta_{(a_i,b_i)}$ *and that* $shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with Berger measure* $\xi_0 = \sum_{i=0}^{n-1} \rho_i \delta_{a_i}$*, where* $0 < \rho_i < 1, a_i \ge 0, b_i > 0$ *for all* $0 \le i \le n-1$ *and* $\sum_{i=0}^{n-1} \rho_i = 1$*. Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if* $\beta_{00}^2 \le (\sum_{i=0}^{n-1} \frac{\rho_i}{b_i})^{-1}$*. In this case, the Berger measure of* $W_{(\alpha,\beta)}$ *is* $\mu = A\ \mu_{\mathcal{M}} + (1-A) \sum_{i=0}^{n-1} \rho_i \delta_{(a_i,0)}$*, where* $A := \beta_{00}^2 \sum_{i=0}^{n-1} \frac{\rho_i}{b_i}$.

Proof. Observe that $||\frac{1}{t}||_{L^1(\mu_{\mathcal{M}})} = \sum_{i=0}^{n-1} \frac{\rho_i}{b_i}$, $(\mu_{\mathcal{M}})_{ext} = \mu_{\mathcal{M}}$ and $(\mu_{\mathcal{M}})_{ext}^X = \xi_0$. Thus, by Lemma 2.2, we have $W_{(\alpha,\beta)}$ is subnormal if and only if $\beta_{00}^2 \leq (\sum_{i=0}^{n-1} \frac{\rho_i}{b_i})^{-1}$. In this case, the Berger measure is given by (2.1). Therefore we have the desired result. □

EXAMPLE 2.9. *If $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with 3-atomic Berger measure $\mu_{\mathcal{M}} = \frac{1}{3}\delta_{(0,\frac{1}{3})} + \frac{1}{3}\delta_{(\frac{1}{3},\frac{1}{2})} + \frac{1}{3}\delta_{(\frac{1}{2},1)}$ and if $shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with Berger measure $\xi_0 = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{3}} + \frac{1}{3}\delta_{\frac{1}{2}}$, then $W_{(\alpha,\beta)}$ is subnormal if and only if $\beta_{00}^2 \leq \frac{1}{2}$. Moreover, by Lemma 2.5, $W_{(\alpha,\beta)}$ is hyponormal if and only if $\beta_{00}^2 \leq \frac{11}{18}$. Therefore, we can see that if $\frac{1}{2} < \beta_{00}^2 \leq \frac{11}{18}$ then $W_{(\alpha,\beta)}$ is hyponormal but not subnormal.*

REMARK 2.10. In [10], Curto, Muhly and Xia conjectured that if $\mathbf{T} = (T_1, T_2)$ is a pair of commuting subnormal operators on $\mathcal{H}$, then the hyponormal condition of $\mathbf{T}$ is enough for the subnormality of $\mathbf{T}$. In many recent papers containing ([6],[7],[8][11],[12],[18]), the conjecture was showed negatively. Example 2.9 is also a simple new counterexample for the Curto-Muhly-Xia conjecture.

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